

A note on the foundation of relativistic mechanics

I: Relativistic observables and relativistic states

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Is there a version of the notions of “state” and “observable” wide enough to apply naturally and in a covariant manner to relativistic systems? I discuss here a tentative answer.

I. INTRODUCTION

The formalism of mechanics and its basic notions, such as “state” and “observable”, are usually defined at first in the *nonrelativistic* context. One then modifies the formalism in order to take gauge invariance into account [1]. Then one deals with reparametrization invariant theories, such as the ones describing relativistic gravitational systems, as special cases of gauge systems in which the gauge includes spacetime transformations. This way of proceeding leads to a number of confusing issues and conceptual difficulties concerning the notions of observable and state, in particular in the gravitational quantum domain. I think that these difficulties are a consequence of the fact that the notions of state and observable defined in a nonrelativistic manner at fixed external time are being stretched and applied forcefully in the relativistic domain, where they do not belong.

Here, I suggest a different approach. I start from scratch, considering definitions of “state” and “observable” given in a naturally relativistic manner. These relativistic notions are introduced in Section II, taking a simple pendulum as an example. These notions are slightly, but significatively distinct from the corresponding conventional nonrelativistic notions. The main difference is essentially that they do not refer to a moment of time. I think that in a relativistic context these relativistic notions of state and observable are more natural to use than the usual ones, which are intrinsically noncovariant. The importance of the relativistic notion of state has been stressed in particular by Dirac [2] and Souriau [3], and certainly by many others. The relativistic notion of (“partial”) observable has been recently discussed for instance in [4].

It is a remarkable fact that the structure of mechanics simplifies drastically when written in terms of the relativistic notions of states and observable. Hamiltonian mechanics takes a particularly elegant and simple form. This general and simple form was noticed and developed in different degrees and with different motivations by many people, starting from Lagrange [5] and including Arnold, Dirac and Souriau. I present it here in Section III in an elementary form, based on the relativistic notions of state and observable.

In Section IV, I present some remarks on general covariant quantum mechanics, although the subject will be treated more in detail elsewhere [6]. In the companion paper [7], I discuss the applications of these ideas to field theory, and in particular in general relativity. I expect the relativistic definition of observables and states discussed here to be especially relevant in quantum gravity, where the conventional notions of position and time are not available.

II. PARTIAL OBSERVABLES AND HEISENBERG STATES

Imagine we want to study empirically the small oscillations of a pendulum. To this aim, we need two measuring devices. A clock and a device that reads the angle of elongation of the pendulum. Let t be the reading of the clock and α the reading of the device measuring the pendulum elongation. Call the variables t and α the *partial observables* [4], or, if there is no ambiguity, simply the *observables* of the system. Let \mathcal{C} be the two-dimensional space with coordinates t and α . Call \mathcal{C} the *extended or relativistic configuration space* of the pendulum, or simply, if there is no ambiguity, the *configuration space* of the pendulum.

We perform a sequence of measurements. We get a sequence of pairs (t, α) . We call each such pair a *correlation* between the two partial observables t and α . Each pair determines a point in the extended configuration space \mathcal{C} . The reason we can do science is the fact that experience shows we can find mathematical relations characterizing the sequences of correlations that are physically realized.

These relations have the following form. We perform a sequence of measurements of t and α , and find that the points representing the measured pairs sit on a curve in the space \mathcal{C} . Let this curve be represented as a relation in \mathcal{C}

$$f(\alpha, t) = 0. \quad (1)$$

Call this curve a *motion* of the system. Thus a motion is a certain relation between observables. We then disturb the pendulum (push it with a finger) and repeat the entire experiment over. At each repetition of the experiment, a different curve is found. That is, a different

mathematical relation of the form (1) is found. However (this is the central point), experience shows that the space of possible curves is limited: indeed, it is a two-dimensional space: there is just a two-dimensional space of curves that are realized in nature.

Denote Γ this two-dimensional space of curves, and let A and ϕ be coordinates on Γ . The two-dimensional space Γ with coordinates A and ϕ is the *space of the motions*, or the *Heisenberg phase space*, or, if there is no ambiguity, simply the *phase space* of the pendulum. We denote a point in Γ as a *motion* of the pendulum, or a *Heisenberg state*, or simply, if there is no ambiguity, a *state*.

The equation that captures the experimental relations in full has therefore the form

$$f(\alpha, t; A, \phi) = 0. \quad (2)$$

That is: for each point in Γ with coordinates (A, ϕ) , we have a curve in \mathcal{C} . Equation (2) is the *evolution equation* of the system.

Concretely, in the case of the (small oscillations of a frictionless) pendulum, the evolution equation is

$$f(\alpha, t; A, \phi) = \alpha - A \sin(\omega t + \phi) = 0. \quad (3)$$

The pair (A, ϕ) labels different sequences of measurements. For each sequence, (A, ϕ) determines a curve in the (t, α) plane, expressing the measured, or predicted, correlation between t and α . Thus a state (a pair (A, ϕ)) determines a motion of the pendulum (a specific relation between t and α) via the evolution equation (3). Each time we disturb the pendulum by interacting with it, or each time we start a new experiment over with the same pendulum, we have a new state. On the other hand, the state remains the same (disregarding quantum theory) if we just observe the pendulum and the clock without disturbing them.

Summarizing: *each state in the phase space determines a relation between the observables in the configuration space*. Each such relation is called a motion. The set of these relations is captured by the evolution equation (2), namely by the vanishing of a function

$$f : \Gamma \times \mathcal{C} \rightarrow R. \quad (4)$$

The evolution equation $f = 0$ expresses all the predictions that can be made using the theory. Equivalently, these predictions are completely captured by fixing the surface (4) in the Cartesian product $\Gamma \times \mathcal{C}$ of the phase space with the configuration space.

Concretely, predictions can be obtained as follows. We first need to perform enough measurements to deduce A and ϕ , namely to find out the state. Once the state is determined or guessed, the evolution equation (3) predicts the allowed correlation between the observables t and α in any subsequent measurement. These predictions are valid until the pendulum is disturbed.

The (\mathcal{C}, Γ, f) structure described above for the example of the pendulum is completely general, and is present in

all relativistic and nonrelativistic fundamental systems. All fundamental systems can be described (at the accuracy at which quantum effects can be disregarded) by making use of these fundamental concepts:

- (i) The *configuration space* \mathcal{C} , of the *observables* of the theory.
- (ii) The *phase space* Γ of the *states* of the theory.
- (iii) The *evolution equation* of the theory $f = 0$, where $f : \Gamma \times \mathcal{C} \rightarrow V$.

V is a (finite or infinite dimensional) vector space. The state in the phase space Γ is fixed until the system is disturbed. Each state in Γ determines (via $f = 0$) a motion of the system, namely a relation (or a set of relations if $\dim(V) > 1$), between the observables in \mathcal{C} . The task of mechanics is to find such a description for all physical systems.

The construction of this description for a given system is conventionally separated in two steps. The first step, kinematics, consists in the specification of the observables that characterize the system. Namely the specification of the configuration space \mathcal{C} and its physical interpretation. (Physical interpretation means the association of certain coordinates on \mathcal{C} with certain measuring devices.) The second step, dynamics, consists in finding the phase space Γ and the function f that describe the correlations in the system.

Notice that the notions of instantaneous state, evolution in time, or observable at fixed time, play no role in the general definitions of observable and state given. This is why these definitions apply naturally to a special and generally relativistic context. Of course, the usual notions can be easily recovered for nonrelativistic systems, as will be discussed in Section III B.

III. RELATIVISTIC HAMILTONIAN SYSTEMS

The relativistic notions of state and observable defined above find their natural home in the relativistic, or presymplectic hamiltonian formalism, which I summarize here.

Virtually all fundamental physical systems can be described by hamiltonian mechanics (I suppose as a consequence of the fact that they are the classical limit of a quantum system). That is, it turns out that once the kinematics is known, namely once \mathcal{C} has been determined, the dynamics (Γ and f) is completely determined by a function H on the cotangent space $\Omega = T^*\mathcal{C}$.

$$H : T^*\mathcal{C} \rightarrow W. \quad (5)$$

where W is a vector space. (The generalization of this structure to field theory is discussed in the companion paper [7].) The pair (\mathcal{C}, H) fully determines the mechanics of the physical system. The pair (\mathcal{C}, H) is a *covariant*

dynamical system. The function H can be called as *the constraints*, or *the relativistic hamiltonian*, or if there is no ambiguity simply as the *hamiltonian*. Remarkably, all fundamental systems in nature seem to have precisely this structure.

There are several equivalent descriptions of the way in which H determines the phase space Γ and f :

1. From (\mathcal{C}, H) to (\mathcal{C}, Γ, f) : Hamilton equations

In coordinates, (\mathcal{C}, H) determines the evolution equations as follows. Let q^a , with $a = 1, \dots, n$, be coordinates on \mathcal{C} , p_a be the corresponding momenta in $\Omega = T^*\mathcal{C}$ and let $H^i(q^a, p_a)$, with $i = 1, \dots, m = \dim(V)$, be the components of H . The motions are given by the m -dimensional surfaces in \mathcal{C} , coordinatized by m parameters τ_i , obtained by solving the Hamilton equations

$$\frac{\partial q^a(\tau_i)}{\partial \tau_i} = \frac{\partial H^i}{\partial p_a}, \quad \frac{\partial p_a(\tau_i)}{\partial \tau_i} = -\frac{\partial H^i}{\partial q^a}, \quad H^i = 0. \quad (6)$$

2. From (\mathcal{C}, H) to (\mathcal{C}, Γ, f) : the geometrical way

Every cotangent space carries the natural symplectic form $d\theta$, where $\theta = p_a dx^a$ is the Poincaré one-form of the cotangent bundle. The equation $H = 0$ defines a surface Σ in $T^*\mathcal{C}$. The restriction ω of $d\theta$ to this surface is a degenerate two-form with null directions. The integral surfaces of these null directions are called the *orbits* of ω on Σ . The phase space Γ is defined as the space of these orbits. Each such orbit projects down from $T^*\mathcal{C}$ to \mathcal{C} to give a subspace of \mathcal{C} , namely a set of relations on \mathcal{C} . Namely a motion. To compute the orbits, we have to integrate the (multi-)vector field X on Σ which is in the kernel of ω , namely which satisfies the equation

$$\omega(X) = 0 \quad (7)$$

Equation (7) is an elegant geometrical way of writing the Hamilton equations (6). Notice that the system is completely defined just by the pair (Σ, ω) : a space and a form over it. The form ω is closed and degenerate, or presymplectic, hence the denomination presymplectic for this formalism. In the companion paper [7], in particular, I will discuss a very simple, compact and elegant formulation of general relativity in terms of a (Σ, ω) pair.

3. From (\mathcal{C}, H) to (\mathcal{C}, Γ, f) : via Hamilton-Jacobi

Consider the (generalized) Hamilton-Jacobi system of m partial differential equations on \mathcal{C}

$$H \left(q^a, \frac{\partial S(q^a)}{\partial q^a} \right) = 0. \quad (8)$$

Let $S(q^a, Q^a)$ be a n -parameter family of (independent, in a suitable sense) solutions. Then pose

$$f^n(q^a, Q^a, P_a) = \frac{\partial S(q^a, Q^a)}{\partial Q^a} - P_a. \quad (9)$$

In general, only $n-k$ of these equations are independent, and only $2(n-k)$ of the constants X^a, P_a are independent. The constants Q^a, P_a coordinatize thus a $2(n-k)$ dimensional space Γ . This is the phase space, and (9) defines f .

This “relativistic”, or “presymplectic” formulation of mechanics is elegant, very well operationally founded, and far more general than the conventional nonrelativistic formulation based on the triple $(\Gamma_{nr}, \omega_{nr}, H_0)$, where $\Gamma_{nr} = T^*Q$ is the space of the initial data, ω_{nr} its symplectic form and H_0 the nonrelativistic hamiltonian. The relation between the two formulations is discussed below.

A. Nonrelativistic systems

For a nonrelativistic systems, the extended configuration space has the structure

$$\mathcal{C} = \mathbb{R} \times Q, \quad (10)$$

and its coordinates $q^a = (t, q^i)$, with $i = 1, \dots, n-1$, are the time variable $t \in \mathbb{R}$ and the physical degrees of freedom variables $q^i \in Q$, where Q is the usual non relativistic configuration space. The cotangent space $\Omega = T^*\mathcal{C}$ has coordinates $(q^a, p_a) = (t, q^i, p_t, p_i)$, and

$$H = p_t + H_0 \quad (11)$$

where H_0 is the nonrelativistic hamiltonian. The surface Σ turns out to be

$$\Sigma = \mathbb{R} \times \Gamma_{nr} \quad (12)$$

where the coordinate on \mathbb{R} is the time t and $\Gamma_{nr} = T^*Q$ is the usual phase space. The restriction of $d\theta$ to this surface has the form

$$\omega = -dH_0 \wedge dt + \omega_{nr}. \quad (13)$$

The evolution equations (6) or (7) define the physical motions in parametrized form $q^a(\tau) = (t(\tau), q^i(\tau))$. Only the dependence of q^i on t determined implicitly by these functions has physical significance: not the explicit dependence of t and q^i on τ . That is, the physics is contained in a curve in \mathcal{C} , not in the way this curve is parametrized.

We can take the vector field X with the form

$$X = \frac{\partial}{\partial t} + X_{nr} \quad (14)$$

where X_{nr} is a vector field on Γ_{nr} . In this case, equation (7) reduces to the equation

$$\omega_{nr}(X_{nr}) = -dH_0, \quad (15)$$

which is the geometric form of the well known nonrelativistic form of the Hamilton equations

$$\frac{\partial q^a(t)}{\partial t} = \frac{\partial H_0}{\partial p_a}, \quad \frac{\partial p_a(t)}{\partial t} = -\frac{\partial H_0}{\partial q^a}. \quad (16)$$

Therefore the nonrelativistic hamiltonian H_0 generates evolution in the time t . More precisely: H determines how the variables in Q are correlated to the variable t . This can be expressed as “how the variables in Q evolve in time”.

B. Discussion

The definition of state and observable given in Section II differ from the one that commonly used in the nonrelativistic context. In the nonrelativistic context the special variable t is chosen to play a peculiar role.

The usual nonrelativistic definition of *state* refers to the properties of a system *at a certain moment of time*. Denote this conventional notion of state as the “instantaneous state”. The space of the instantaneous states is the usual nonrelativistic phase space Γ_{nr} . For instance, we fix the value $t = t_0$ of the time variable, and characterize the instantaneous state in terms of the initial data, for the pendulum position and momentum (α_0, p_0) , at $t = t_0$. Then (α_0, p_0) are coordinates on Γ_{nr} .

Instead, in Section II I have defined a relativistic state as a solution of the equation of motion. (This is true for a system without gauge invariance. If there is gauge invariance, a state is a gauge equivalence class of solutions of the equations of motion). Thus, the relativistic phase space Γ defined above is the space of the solutions of the equations of motion.

Once a value t_0 of the time has been fixed, there is a one-to-one correspondence between initial data and solutions of the equations of motion. Indeed, each solution of the equation of motion determines initial data at t_0 ; and each choice of initial data at t_0 determines uniquely a solution of the equations of motion. Therefore there is a one-to-one correspondence between instantaneous states and relativistic states. (This one-to-one correspondence is also present if there is gauge invariance, because a choice of initial data determines uniquely a gauge equivalence class of solutions.) The conventional phase space of the initial data, Γ_{nr} , with coordinates (α_0, p_0) is therefore isomorphic with the relativistic phase space Γ , with coordinates (A, ϕ) . The two be can be identified after having chosen a value $t = t_0$ of the time variable. The identification map is given in the example of the pendulum by

$$(A, \phi) \longmapsto (\alpha_0(A, \phi), p_0(A, \phi)) : \quad (17)$$

$$\alpha_0(A, \phi) = A \sin(\omega t_0 + \phi), \quad (18)$$

$$p_0(A, \phi) = \omega m A \cos(\omega t_0 + \phi). \quad (19)$$

Thus (α_0, p_0) can be simply seen as coordinates on Γ , and mathematically the phase space Γ defined above is essentially the same space as the nonrelativistic phase space: $\Gamma \sim \Gamma_{nr}$. But the physical interpretation of the two is quite different: a point of Γ is not seen as representing the instantaneous state of the pendulum, but rather as representing the full history of the pendulum that evolves from that instantaneous state.

An important remark is that in a nonrelativistic system the space of the orbits Γ is in one to one correspondence with the cotangent space Γ_{nr} . Therefore the cotangent space $\Gamma_{nr} = T^*Q$ is the “natural arena” for nonrelativistic hamiltonian mechanics and also the space of the motions. The phase space plays therefore a double role in nonrelativistic hamiltonian mechanics: as the arena of hamiltonian mechanics and as the space of the states. In the relativistic context, in general, this double role is lost: one must distinguish the cotangent space over which hamiltonian mechanics is defined ($\Omega = T^*\mathcal{C}$) from the space of the space of the states Γ (which we call phase space). This distinction becomes important in field theory, where the space on which hamiltonian mechanics is formulated can be finite dimensional even if the phase space is infinite dimensional [7].

In a nonrelativistic system, X_{nr} generates a one-parameter group of transformation in Γ : this sends a state to a state in which the observables are the same, except for t which is changed. This transformation group is the hamiltonian flow of H_0 on Γ . Instead of having the observables in Q depending on t , one can shift perspective and view the observables in Q as time independent objects and the states in Γ as time dependent objects. This is a classical analog of the shift from the Heisenberg to the Shrödinger picture in quantum theory, and can be called the “classical Schrödinger picture”.

On the other hand, in a general relativistic system there is no special “time” variable, \mathcal{C} does not split naturally as $\mathcal{C} = \mathbb{R} \times Q$, the constraints do not have the form $H = p_t + H_0$ and the description of the correlations in terms of “how the variables in Q evolve in time” is not available in general. For these systems the classical Schrödinger picture, in which states evolve in time, is not available: only the relativistic notions of state and observable considered here (partial observables and Heisenberg states) make sense.

For these systems, dynamics is not a theory of the evolution of observable quantities in time. It is the theory of the correlations between partial observables.

C. Simple examples

1. Pendulum

On $T^*\mathcal{C}$ we can put canonical coordinates (t, α, p_t, p) . The function H is (with mass=1)

$$H(t, \alpha, p_t, p_\alpha) = p_t + \frac{p^2 + \omega^2 \alpha^2}{2} \equiv p_t + H_0(\alpha, p). \quad (20)$$

Equations (6) give

$$\frac{dt(\tau)}{d\tau} = 1, \quad \frac{d\alpha(\tau)}{d\tau} = p(\tau), \quad \frac{dp(\tau)}{d\tau} = -\omega^2 \alpha(\tau). \quad (21)$$

whose solutions project down to the \mathcal{C} space to the trajectories

$$f(\alpha, t; A, \phi) = \alpha - A \sin(\omega t + \phi) = 0. \quad (22)$$

Thus the phase space Γ has coordinates A and ϕ and we recover (3) for f .

2. Relativistic particle

The configuration space \mathcal{C} is Minkowski space \mathcal{M} , with coordinates x^μ . The dynamics is given by the hamiltonian $H = p^2 - m^2$, which defines the mass- m Lorentz hyperboloid \mathcal{K}_m . The constraint surface Σ is therefore given by $\Sigma = \mathcal{M} \times \mathcal{K}_m$. The null vectors of the restriction of $d\theta = dp_\mu \wedge dx^\mu$ to Σ are therefore

$$X = p_\mu \frac{\partial}{\partial x^\mu}, \quad (23)$$

because $\omega(X) = p^\mu dp_\mu = 2d(p^2) = 0$ on $p^2 = m^2$. The integral lines of X are

$$x^\mu(\tau) = p^\mu \tau + x_0^\mu \quad (24)$$

which give the physical motions of the particle. The space of these lines is six dimensional (as $p^2 = m^2$ and (p^μ, x_0^μ) defines the same line as $(p^\mu, x_0^\mu + p^\mu a)$ for any a), and represents the phase space. The motions are thus the timelike straight lines in \mathcal{M} .

Notice that all notions used are completely Lorentz invariant. A state is a timelike geodesic; an observable is any Minkowski coordinate, a correlation is a point in Minkowski space. The theory is about correlations between Minkowski coordinates, that is, observations of the particle at a certain spacetime point. On the other hand, the split $\mathcal{M} = \mathcal{R} \times \mathcal{Q}$ necessary to define the usual hamiltonian formalism, is observer dependent.

3. Cosmological model

Consider a cosmological model in which the sole dynamical variables are the radius a of a maximally symmetric universe and the spatially constant value ϕ of a scalar field. Then \mathcal{C} has coordinates a and ϕ . The dynamics is given by a single constraint

$$H(a, \phi, p_a, p_\phi) = 0. \quad (25)$$

The constraint surface has dimension 3, the phase space has dimension 2, and the motions are curves in the (a, ϕ) space. For each state, the theory predicts the correlations between a and ϕ .

D. Two distinct physical meanings of the Lagrangian evolution parameter

Consider the Lagrangian formulation of the dynamics of a pendulum with a single Lagrangian variable $\alpha(t)$. Then the Lagrangian evolution parameter t is one of the configuration space variables.

On the other hand, consider the Lagrangian formulation of the cosmological model discussed above, in which the Lagrangian variables are $a(t)$ and $\phi(t)$ and the action is reparametrization invariant. Or, similarly, consider the reparametrization invariant Lagrangian for a free relativistic particle

$$S = m \int dt \sqrt{(dx^\mu/dt)(dx_\mu/dt)} \quad (26)$$

for the four Lagrangian variables $x^\mu(t)$. In these cases, the Lagrangian evolution parameter t is unphysical, in the sense that it has nothing to do with observability.

One should therefore not confuse the t in the first case with the t in the second case. They have very different physical interpretation. The fact that they are often denoted with the same letter and with the same name is only a very unfortunate historical accident.

Now, suppose we are given a time reparametrization invariant Lagrangian, such as (26) and we blindly Legendre transform to the relativistic form of hamiltonian mechanics, based the *five dimensional* extended configuration space $\tilde{\mathcal{C}} = R \times \mathcal{M}$ with coordinates $q^a = (t, x^\mu)$. (Or (t, a, ϕ) for the cosmological model.) How would we realize that $\tilde{\mathcal{C}}$ has a redundant dimension? The answer is that we find two constraints: the constraint (11) with $H_0 = p^\mu p_\mu - m^2$, and also the constraint

$$H_0 = 0. \quad (27)$$

These imply

$$p_t = 0. \quad (28)$$

which, in turn, gives the evolution equation $\frac{dt}{d\tau} = 0$ for the “time” parameter t . In other words, the Lagrangian parameter t drops out from the relativistic hamiltonian formalism.

More precisely, the constraint surface is $\Sigma = R \times \mathcal{M} \times \mathcal{K}_m$, but the two-form ω , which is given by the restriction of $d\theta = dp_a \wedge dq^a = dp_t \wedge dt + dp_\mu \wedge dx^\mu$ to Σ , knows nothing about t , because the $dp_t \wedge dt$ term vanishes on Σ . Therefore, the variable t decouples completely from the formalism. This fact signals that the “true” extended configuration space \mathcal{C} is not $\tilde{\mathcal{C}} = R \times \mathcal{M}$, but rather $\mathcal{C} = \mathcal{M}$ alone. The observables quantities are genuinely independent from the Lagrangian evolution parameter t .

The partial observables of the relativistic particle are the quantities x^μ , in any Lorentz frame. They represent clocks and devices measuring the spatial position. The role of t is reduced to the one of an arbitrary parameter along the orbits and must not be confused with the role

of the time variable t in a nonrelativistic system, which is a partial observable. The same happens in general relativity for the *four* general relativistic coordinates [7].

IV. QUANTUM MECHANICS

I conclude with a remark on general covariant quantum mechanics, although the subject deserves a more complete discussion, which will be given elsewhere. Consider a small region \mathcal{R} in \mathcal{C} . The region can be taken to represent a certain correlation with a certain associated experimental resolution. For a particle \mathcal{R} can be a small spacetime region in Minkowski space. Fix a state in Γ . The motion determines by the state can either intersect \mathcal{R} or not. Therefore a classical state assigns a yes/no value to each region \mathcal{R} in \mathcal{C} . This yes/no value can be seen as the prediction of whether or not a set of partial observable measuring devices can give a certain (simultaneous) set of values. In the case of the particle, whether or not a detector in the spacetime region \mathcal{R} will detect the particle.

In quantum theory, the predictions of the theory are not deterministic, but probabilistic. Accordingly, a quantum state associates a probability amplitude $P_{\mathcal{R}}$ to every small region of \mathcal{C} . How are these probabilities computed?

The quantum dynamics of a system is entirely characterized by a complex two-point function on the configuration space

$$W : \mathcal{C} \times \mathcal{C} \rightarrow C. \quad (29)$$

This function is called the *propagator*. In the classical limit, the propagator is approximated by the exponential of a solution of the Hamilton-Jacobi system consider above.

$$W(x^i, X^i) \sim e^{\frac{i}{\hbar} S(x^i, X^i)}. \quad (30)$$

The exact propagator satisfies the Schrödinger system

$$C \left(x^i, \frac{\partial}{\partial x^i} \right) W(x^i, X^i) = 0. \quad (31)$$

The Hilbert space of the theory can be constructed from W as follows. Start form a space \mathcal{E} of functions $f(x)$ on \mathcal{C} (say smooth compact support). Consider the bilinear form on \mathcal{E} whose kernel is $W(x, x')$

$$\langle f | f' \rangle \equiv \int dx dx' \overline{f(x)} W(x, x') f(x'). \quad (32)$$

The Hilbert space \mathcal{H} of the theory is obtained by equipping \mathcal{E} with this scalar product, dividing by the zero norm subspace and completing in norm. The map

$$\begin{aligned} P : \mathcal{E} &\rightarrow \mathcal{H} \\ f &\mapsto |f\rangle \end{aligned} \quad (33)$$

is highly degenerate, and is sometime (improperly) called the “projector”.

To each region \mathcal{R} in \mathcal{C} , we can associate the state $|\mathcal{R}\rangle = C_{\mathcal{R}} |f_{\mathcal{R}}\rangle$, where $f_{\mathcal{R}}$ is the characteristic function of \mathcal{R} and $C_{\mathcal{R}} = \langle f_{\mathcal{R}} | f_{\mathcal{R}} \rangle^{-1/2}$ is the normalization. For a region \mathcal{R} sufficiently small (smaller than any other quantity in the problem), the probabilities that express all predictions of quantum theory are defined by

$$P_{\mathcal{R}} = |\langle \mathcal{R} | \Psi \rangle|^2. \quad (34)$$

This interpretation postulate reduces to the well known interpretation of the modulus square of the wave function as spatial probability density for nonrelativistic systems [6]. After the correlation determined by \mathcal{R} has been verified, the state of the system is $|\mathcal{R}\rangle$.

In particular, the quantity

$$A_{\mathcal{R}, \mathcal{R}'} = \langle \mathcal{R} | \mathcal{R}' \rangle \quad (35)$$

is the probability amplitude to detect the system in the (small) region \mathcal{R} of the extended configuration space, if the system was previously detected in the (small) region \mathcal{R}' . This amplitude can be written explicitly in terms of the propagator

$$\begin{aligned} A_{\mathcal{R}, \mathcal{R}'} &= C_{\mathcal{R}} C_{\mathcal{R}'} \int_{\mathcal{R}} dx \int_{\mathcal{R}'} dy W(x, y) \\ &= \frac{\int_{\mathcal{R}} dx \int_{\mathcal{R}'} dy W(x, y)}{\sqrt{\int_{\mathcal{R}} dx \int_{\mathcal{R}} dy W(x, y)} \sqrt{\int_{\mathcal{R}'} dx \int_{\mathcal{R}'} dy W(x, y)}}. \end{aligned}$$

See [6] and references therein. On covariant approaches to quantum theory see also [8] and references therein.

The important point I want to stress here is that quantum theory as well can be formulated in a fully covariant language in which time plays no special role. The fundamental ingredient is once more the extended configuration space of the partial observables \mathcal{C} . Quantum theory gives probabilities of observing a certain correlation given that a certain other correlation was been observed.

V. CONCLUSION

The difference between the relativistic definition of state and observable that we have studied here and the nonrelativistic definition is the role played by time. In the nonrelativistic context time is a primary concept. Mechanics is defined as the theory of the evolution in time. In the definition considered here, on the other hand, there is no special partial observable singled out as the independent variable. Mechanics is defined as the theory of the correlations between partial observables – the time variable may be just one among these.

Historically, the idea that the time independent notion of state is needed in a relativistic context has been advocated particularly by Dirac [2] and by Souriau [3].

The advantages of the relativistic notion of state are multifold. In special relativity time transforms with other variables, and there is no covariant definition of instantaneous state. In a Lorentz invariant field theory, in particular, the notion of instantaneous state breaks explicit Lorentz covariance: The instantaneous state is the value of the field on a simultaneity surface, which is such for a certain observer only. The notion of Heisenberg state, on the other hand, is Lorentz invariant.

This shift in perspective, however, is forced in general relativity, where the notion of a special spacelike surface over which initial data are fixed conflicts with diffeomorphism invariance. A generally covariant notion of instantaneous state, or a generally covariant notion of observable at a given time, make very little physical sense. Indeed, none of the various notions of time that appear in general relativity (coordinate time, proper time, clock time) can play the full role that t plays in nonrelativistic mechanics. A consistent definition of state and observable in a generally covariant context cannot explicitly involve time.

Physically, the reason of this difference is simple. In nonrelativistic physics, time and spacial position are defined with respect to a system of reference bodies and clocks that are always implicitly assumed to exist and not to interact with the physical system studied. In gravitational physics, one discovers that no body or clock exists which does not interact with the gravitational field: the gravitational field affects directly the motion and the rate of any reference body or clock. Therefore one cannot separate reference bodies and clocks from the dynamical variables of the system. General relativity, or any other general covariant theory, is always a theory of interacting variables that necessarily include the bodies of clocks used as references to characterize spacetime points. In the example of the pendulum discussed above we can assume that the pendulum itself and the clock do not interact. In a general relativistic context the two always interact, and therefore \mathcal{C} does not split cleanly into Q and \mathbb{R} .

Mechanics can be seen as the theory of the evolution of the physical variables in time only in the nonrelativistic limit. In a fully relativistic context, mechanics is a theory of correlations between partial observables, or the theory of the relative evolution of partial observables with respect to each other.

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